

Generalized damped Milne-Pinney equation and Chiellini method

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Abstract

We adopt the Chiellini integrability method to find the solutions of various generalizations of the damped Milne-Pinney equations. In particular, we find the solution of the damped Ermakov-Painlevé II equation and generalized dissipative Milne-Pinney equation.

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1 Introduction

Ermakov [1] used the Milne Pinney equation [2], [3] while investigating a first integral for the corresponding time dependent harmonic oscillator. Since then, this nonlinear equation

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has gained intensive attention [4], [5] of physicists and engineers due to its widespread application in many physical problems such as propagation of laser beams in nonlinear media, plasma dynamics etc. It is well known that the general solution for the Milne-Ermakov-Pinney equation

$$\ddot{y} + \omega^2(t)y = \frac{\kappa}{y^3}, \quad (1.1)$$

where \ddot{y} denotes double differentiation of y with respect to time t , $\omega = \omega(t)$ is a time-dependent frequency function and κ is a numerical constant can be written as $y = (Ax_1^2 + 2Bx_1x_2 + Cx_2^2)^{1/2}$, where A, B and C are constants such that $AC - B^2 = \kappa$ and x_1 and x_2 are two independent solutions for the time-dependent harmonic oscillator equation $\ddot{x} + \omega^2(t)x = 0$.

Equation (1.1) does not include any mechanism of damping. Hence it is natural to add a term linear in the velocity, yielding the damped Milne-Ermakov-Pinney equation

$$\ddot{y} + \mu\dot{y} + \omega^2(t)y = \frac{\kappa}{y^3}, \quad (1.2)$$

where $\mu > 0$ is a constant positive parameter. Equation (1.2) can be transformed into generalized Emden-Fowler equation of index -3 which satisfies integrability.

In recent times a hybrid Ermakov-Painlevé II system was derived by Rogers [6] in a pioneering work as a reduction of a coupled $N+1$ -dimensional Manakov-type NLS system. He showed that the Ermakov invariants admitted by the hybrid system were key to its systematic reduction in terms of a single component Ermakov-Painlevé II equation which, in turn, may be linked to the integrable Painlevé II equation.

The application of the Chiellini integrability condition to find the solutions of nonlinear differential equations has been recently promoted by two groups; Harko, Mak and their coauthors [15, 16, 17, 19] and Mancas and Rosu [20, 21, 22]. It must be worth to note that the Chiellini integrability condition appears quite naturally for Hamiltonization of the Liénard equation using the Jacobi multiplier technique. Our intention in this letter is to extend the scope of the integrability condition given by Chiellini to find out solutions to new kind of damped Ermakov-Milne-Pinney systems. In particular, we obtain the analytic solutions of the damped Ermakov-Painlevé II equation and generalized damped Milne-Pinney equation.

The main result of this paper is given as follows.

Proposition 1.1 (a) *Let the damped Ermakov-Painlevé II equation*

$$\ddot{y} + g(y)\dot{y} + h(y) = 0, \quad h(y) = \lambda y + \epsilon y^3 - \frac{\eta}{y^3},$$

satisfies the Chiellini integrability condition $\frac{d}{dy}(\frac{h(y)}{g(y)}) = pg(y)$, for some constant p . The solution of the above equation is given by

$$y = \sqrt{\frac{1}{(t-t_0)^2} + \sqrt{\frac{c}{3}}}, \quad \epsilon = -1 \quad y = \sqrt{\left(\frac{c^2}{16\lambda^2} - \frac{\eta}{\lambda}\right)^{\frac{1}{2}} \sin[2\sqrt{2\lambda}(t-t_0)] + \frac{c}{4\lambda}}, \quad \epsilon = 0.$$

(b) If the generalized damped Milne-Pinney equation

$$\ddot{y} + g(y)\dot{y} + \lambda y = \frac{k_1}{y^3} + \frac{k_2}{y^2} + \sum_{n=0}^R \delta_n y^{2n+1}$$

satisfies Chiellini condition then a parametric solution of this equation for $R = 0$ is given by

$$t = y_0 \omega + \frac{f'(y_0)}{4\wp'(\omega_0)} \left[\log \frac{\sigma(\omega + \hat{c} - \omega_0)}{\sigma(\omega + \hat{c} + \omega_0)} + 2(\omega + \hat{c})\zeta(\omega_0) \right] + \delta$$

$$y = y_0 + \frac{f'(y_0)}{4[\wp(\omega + \hat{c}) - \frac{f''(y_0)}{24}]}$$

where δ is an integrating constant and \hat{c} being any fixed constant.

Also $f(y) = 2(\delta_0 - \lambda)y^4 + cy^2 - 4k_2y - 2k_1$, y_0 is a root of the equation $f(y) = 0$ and $\wp(\omega)$ is the Weierstrass \wp - function

Rest of the article is devoted to the proof of our main result.

2 Chiellini method and solution of equations

The first order Abel differential equation [7] of the first kind plays an important role in many physical and mathematical problems. The connection between the second-order nonlinear differential equations and the Abel equation is well known [8],[9] and the solutions to such differential equations can often be obtained via the solutions of the corresponding Abel differential equations. A second order differential equation of the Liénard type [10] given by

$$\ddot{Y} + g(Y)\dot{Y} + h(Y) = 0 \quad (2.1)$$

may be transformed into a first-order Abel differential equation of second kind, namely

$$z \frac{dz}{dY} + g(Y)z + h(Y) = 0 \quad (2.2)$$

by the transformation $\dot{Y} = z(Y(t))$, which in turn is transformed to the Abel equation of first kind

$$\frac{dX}{dY} = g(Y)X^2 + h(Y)X^3 \quad (2.3)$$

via the transformation $z = \frac{1}{X}$. However, the criterion of integrability of such equations greatly depends on the expressions of $g(Y)$ and $h(Y)$. An important observation by Chiellini [11] in 1931 states that a first kind Abel differential equation (2.3) is exactly integrable if the functions $g(Y)$ and $h(Y)$ satisfies the condition

$$\frac{d}{dY} \left(\frac{h(Y)}{g(Y)} \right) = pg(Y) \quad (2.4)$$

for some constant p . This integrability condition has been applied in 1960s by Bandić who wrote a couple of mathematical papers [12], [13] and then by Borghero and Melis [14] in the Szebehely's problem. Recently, this integrability condition has gained much attention by Mak and Harko [15], [16], [17] in obtaining general solutions of the first-kind Abel equations from a particular solution. This result has also been used by Yurov and Yurov [18] in cosmology and again by Harko et al [19] in case of particular Liénard equations.

The Chiellini condition not only ensures the integrability of a system but it also helps to find the solution. If we further require that $z = c_k \frac{h(Y)}{g(Y)}$ then its substitution in equation (2.2) leads to

$$pc_k^2 + c_k + 1 = 0 \Rightarrow c_k = \frac{-1 \pm \sqrt{1 - 4p}}{2p}.$$

For simplicity we choose $c_k = 1$ which gives the value of $p = -2$ in equation (2.4). Thus from $\dot{Y} = z(Y(t))$ we have

$$\dot{Y} = \frac{h(Y)}{g(Y)}. \quad (2.5)$$

Using this result we arrive at a much relevant observation that equation (2.1) can be turned to the non dissipative equation

$$\ddot{Y} + H(Y) = 0, \quad H(Y) = 2h(Y) \quad (2.6)$$

where the function $h(Y)$ is scaled up by a factor 2.

This result allows us to find the dissipation function in (2.1) without actually knowing Y . Multiplying $\ddot{Y} + 2h(Y) = 0$ by \dot{Y} and integrating we have

$$\dot{Y}\ddot{Y} + 2\dot{Y}h(Y) = 0 \Rightarrow \dot{Y}^2 = -4 \int h(Y)dY + c \quad (2.7)$$

where c is an integrating constant.

Thus from (2.5) we have

$$g(Y) = \frac{h(Y)}{\sqrt{c - 4 \int h(Y)dY}} \quad (2.8)$$

Upon further integration of (2.7) we have

$$t - t_0 = \int \frac{dY}{\sqrt{c - 4 \int h(Y)dY}} \quad (2.9)$$

where t_0 depends on an initial condition.

3 Solutions of Dissipative Ermakov-Painlevé II and generalized Milne-Pinney equations

Combining the terms of both Ermakov-Pinney equation and the Painlevé II we obtain the following equation

$$\ddot{y} + \frac{\tau}{2}y + \epsilon y^3 = -\frac{1}{4y^3}(\gamma - \frac{\epsilon}{2})^2. \quad (3.1)$$

This nonlinear equation is known as the (single component) Ermakov-Painlevé II equation and was derived by Rogers *et al* [23, 6]. It is related [24] to the Painlevé II equation

$$\ddot{z} = 2z^3 + \tau z + \gamma, \quad (3.2)$$

where

$$z = \frac{\epsilon}{2y^2}(\gamma - \frac{\epsilon}{2} - 2y\dot{y}).$$

If we express $y = \sqrt{|\phi|^2 + \psi|^2}$ then the canonical single component Ermakov-Painlevé II equation yields a particular Ermakov-Ray-Reid system (for details, see [6]) and admits the characteristic invariant which may be exploited systematically to construct the solutions.

Let us assume the equation (3.1) as

$$\ddot{y} + \lambda y + \epsilon y^3 = \frac{\eta}{y^3} \quad (3.3)$$

where $\lambda = \frac{\tau}{2}$ and $\eta = -\frac{1}{4}(\gamma - \frac{\epsilon}{2})^2$

If

$$h(y) = \lambda y + \epsilon y^3 - \frac{\eta}{y^3} \quad (3.4)$$

then equation (3.3) can be written as

$$\ddot{y} + h(y) = 0 \quad (3.5)$$

We introduce the dissipative Ermakov-Painlevé II equation having same $h(y)$ as in the non dissipative case but with an additional damping term. The equation

$$\ddot{Y} + g(Y)\dot{Y} + h(Y) = 0 \quad (3.6)$$

is called Chiellini dissipative Ermakov Painlevé II equation because the damping coefficient $g(Y)$ will be obtained from the Chiellini integrability condition. Using $h(Y)$ from (3.4) in (2.7) we have

$$\dot{Y} = \sqrt{c - \epsilon Y^4 - 2\lambda Y^2 - 2\eta Y^{-2}} \quad (3.7)$$

and in (2.8) we have

$$g(Y) = \frac{\lambda Y^2 + \epsilon Y^4 - \eta Y^{-2}}{\sqrt{-\epsilon Y^6 - 2\lambda Y^4 + cY^2 - 2\eta}} \quad (3.8)$$

Further from (3.7) upon integration once more we have

$$\begin{aligned} Y^2 &= \frac{1}{(t-t_0)^2} + \sqrt{\frac{c}{3}}, & \epsilon &= -1 \\ &= \sqrt{\frac{c^2}{16\lambda^2} - \frac{\eta}{\lambda}}^{\frac{1}{2}} \sin[2\sqrt{2\lambda}(t-t_0)] + \frac{c}{4\lambda}, & \epsilon &= 0 \end{aligned} \quad (3.9)$$

t_0 depending on initial conditions.

3.1 Generalized dissipative Milne-Pinney equation

At first we embark a simple equation of this category and obtain its solution. Let us consider the equation

$$\ddot{Y} + g(Y)\dot{Y} - \frac{\delta}{Y^5} = 0 \quad (3.10)$$

This may be written as equation(2.1) with $h(Y) = -\frac{\delta}{Y^5}$.

Following similar arguments we have

$$\dot{Y} = \sqrt{c - \delta Y^{-4}}. \quad (3.11)$$

The dissipative term

$$g(Y) = -\frac{\delta}{Y^3 \sqrt{cY^4 - \delta}}.$$

and a parametric solution to equation (3.11) in terms of Weierstrass \wp function is given as

$$\begin{aligned} t &= y_0^2 \omega + \left[-\frac{y_0 f'(y_0)}{2\wp'(\omega_0)} + \frac{f'(y_0)^2 \wp''(\omega_0)}{16\wp'(\omega_0)^3} \right] \log \frac{\sigma(\omega + \hat{c} + \omega_0)}{\sigma(\omega + \hat{c} - \omega_0)} \\ &\quad - \frac{f'(y_0)^2}{16\wp'(\omega_0)^2} [\zeta(\omega + \hat{c} + \omega_0) + \zeta(\omega + \hat{c} - \omega_0)] \\ &\quad + (\omega + \hat{c}) \left(\frac{y_0 f'(y_0)}{\wp'(\omega_0)} \zeta(\omega_0) - \frac{f'(y_0)^2}{16} \left[\frac{2\wp(\omega_0)}{\wp'(\omega_0)^2} + \frac{2\wp''(\omega_0)\zeta(\omega_0)}{\wp'(\omega_0)^3} \right] \right) + \hat{\delta} \end{aligned} \quad (3.12)$$

and

$$Y = y_0 + \frac{f'(y_0)}{4[\wp(\omega + \hat{c}) - \frac{f''(y_0)}{24}]} \quad (3.13)$$

where $\hat{\delta}$ is an integrating constant and \hat{c} being any fixed constant.

Also $f(Y) = cY^4 - \delta$, y_0 is a root of the equation $f(Y) = 0$ and $\rho(\omega) = \rho(\omega, g_2, g_3)$ is the Weierstrass ρ - function attached to the Weierstrass Invariants. Here $g_2 = 3\alpha_2^2 - 4\alpha_1\alpha_3$ and $g_3 = 2\alpha_1\alpha_2\alpha_3 - \alpha_2^3 - \alpha_0\alpha_3^2$; $\alpha_0 = c$, $\alpha_1 = cy_0$, $\alpha_2 = cy_0^2$, $\alpha_3 = cy_0^3$. $\sigma(\omega)$ and $\eta(\omega)$ are the Weierstrass sigma and Weierstrass zeta functions respectively, $\wp(\omega_0) = \frac{f''(y_0)}{24}$ (for a choice of ω_0). Expression (3.12) is obtained from formula 1037.11 in [25].

Generalizing the results of equation (2.1) we consider the following example of generalized damped Milne-Pinney equation. Let us consider the equation

$$\ddot{Y} + g(Y)\dot{Y} + \lambda Y = \frac{k_1}{Y^3} + \frac{k_2}{Y^2} + \sum_{n=0}^R \delta_n Y^{2n+1} \quad (3.14)$$

For $R = 0$ we have

$$\dot{Y} = \sqrt{2(\delta_0 - \lambda)Y^2 + c - 4k_2Y^{-1} - 2k_1Y^{-2}} \quad (3.15)$$

The dissipative term

$$g(Y) = \frac{(\lambda - \delta_0)Y - k_2Y^{-2} - k_1Y^{-3}}{\sqrt{2(\delta_0 - \lambda)Y^2 + c - 4k_2Y^{-1} - 2k_1Y^{-2}}}$$

Further integration of (3.15) yields a parametric solution in terms of Weierstrass \wp function given as

$$t = y_0\omega + \frac{f'(y_0)}{4\wp'(\omega_0)} \left[\log \frac{\sigma(\omega + \hat{c} - \omega_0)}{\sigma(\omega + \hat{c} + \omega_0)} + 2(\omega + \hat{c})\zeta(\omega_0) \right] + \delta \quad (3.16)$$

$$Y = y_0 + \frac{f'(y_0)}{4[\wp(\omega + \hat{c}) - \frac{f''(y_0)}{24}]} \quad (3.17)$$

where δ is an integrating constant and \hat{c} being any fixed constant.

Also $f(Y) = 2(\delta_0 - \lambda)Y^4 + cY^2 - 4k_2Y - 2k_1$, y_0 is a root of the equation $f(Y) = 0$ and $\wp(\omega) = \wp(\omega, g_2, g_3)$ is the *Weierstrass \wp -function*. Here $g_2 = 3\alpha_2^2 - 4\alpha_1\alpha_3$ and $g_3 = 2\alpha_1\alpha_2\alpha_3 - \alpha_2^3 - \alpha_0\alpha_3^2$;

$$\alpha_0 = 2(\delta_0 - \lambda)$$

$$\alpha_1 = 2(\delta_0 - \lambda)y_0$$

$$\alpha_2 = 2(\delta_0 - \lambda)y_0^2 + \frac{c}{6}$$

$$\alpha_3 = 2(\delta_0 - \lambda)y_0^3 + 3\frac{cy_0}{6} - k_2$$

$\wp(\omega_0) = \frac{f''(y_0)}{24}$ (for a choice of ω_0) is not equal to any of the roots of $4y^3 - g_2y - g_3 = 0$. Expression (3.16) is obtained from formula 1037.06 in [25].

4 Conclusion

In this letter we have computed the solutions to a class of exactly integrable generalized damped Milne-Pinney equations. If the coefficients of the second-order nonlinear equations satisfy some specific conditions that follow from the Chiellini integrability, then the general solution of the damped Milne-Pinney equation can be obtained in an exact parametric form. In particular, we have obtained the solutions of the damped Ermakov-Painlevé II and another generalized damped Milne-Pinney equation.

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